

Manipulation of Dirac Cones in Mechanical Graphene

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(Dated: May 26, 2015)

Mechanical graphene, which is a spring-mass model with the honeycomb structure, is investigated. The vibration spectrum is dramatically changed by controlling only one parameter, spring tension at equilibrium. In the spectrum, there always exist Dirac cones at K- and K'-points. As the tension is modified, extra Dirac cones are created and annihilated in pairs. When the time reversal symmetry is broken by uniform rotation of the system, creation and annihilation of the Dirac cones result in a jump of the appropriately defined Chern number. Then, a flip of the propagation direction of the chiral edge modes takes place, which gives an experimental way to detect the topological transition. This is a bulk-edge correspondence of the classical system. We also demonstrate the other important concept, symmetry protection of the topological states, is at work in the classical system. For the time reversal invariant case, the topological edge modes exist for the fixed boundary condition but not for the free boundary condition. This contrast originates from the symmetry breaking at the free boundary.

PACS numbers: 63.22.Rc, 03.65.Vf, 45.20.D-

I. INTRODUCTION

Graphene [1], a two-dimensional crystal with a honeycomb array of carbon atoms, has been one of the most hot topics in condensed matter physics in this decade. The most prominent feature of graphene is existence of massless Dirac fermions, or Dirac cones at the Fermi energy. In general, not limited to graphene, Dirac cones allow a lot of properties distinct from usual metals or semiconductors, such as unique responses to electronic and magnetic field [1, 2], or characteristic edge states [3, 4]. One interesting subject of study is a manipulation of Dirac cones. For instance, shifting the Dirac cones in momentum space induces a gauge field without breaking the time reversal symmetry [5]. By inducing a mass of Dirac cones (gap opening), we have a chance to observe topologically nontrivial phases. For instance, by appropriately breaking the time reversal symmetry, graphene can go into a quantum Hall state without external magnetic field [6]. Or, if the spin degrees of freedom and the spin-orbit coupling are explicitly taken account of, graphene becomes a time reversal invariant Z_2 topological insulator [7].

One direction of recent developments in Dirac and topological systems is exporting the concept to systems other than conventional solid materials. For instance, manipulation of Dirac cones is experimentally realized in an optical lattice cold atoms [8]. The other example is electromagnetic field in a photonic crystal, which is governed by the classical Maxwell equation. As for triangular and honeycomb lattices, the Dirac cones in photonic band are demonstrated [9–12]. Furthermore, topologically nontrivial phase is achieved in photonic crystals or

a coupled resonator system [13–15]. Very recently, other classical systems, i.e., mechanical systems obeying the Newton's equation of motion are also discussed in the context of Dirac cones or topological edge states [16–24]. One great advantage of these kinds of artificial systems is their controllability, which enables us to access parameter region unreachable in solids. We can easily realize phenomena that are difficult in solids such as merging of Dirac cones [25].

In this paper, a honeycomb spring-mass model [26], we dubbed it as mechanical graphene, is investigated as a typical mechanical system in which Dirac physics plays a role. We propose a simple and feasible way, just stretching the system uniformly and isotropically, to manipulate Dirac cones. The uniform stretch modifies tension of springs at equilibrium, which controls the relation between the transverse and longitudinal waves. Then, the frequency spectrum varies as a function of the tension. As the tension grows from zero, the frequency band structure first looks like the one in p -orbital honeycomb optical lattice [27] (phase I), then becomes like the one in bilayer graphene with the trigonal warping [28] (phase II). Finally the spectrum is given by that of graphene (phase III), but with extra degeneracy. Merging of the Dirac cones is observed at the two critical points. This topological transition associated with the merging of the Dirac cones is detected as a jump in the appropriately defined Chern number [21, 29], when the time reversal symmetry is broken by uniform rotation of the system. The change in the Chern number is also observed by the propagation direction of the “chiral edge modes” [30, 31]. Not only proposing a practical method for Dirac cone manipulation, we also demonstrate the state-of-art idea of topological phenomena, symmetry protection, is at work in this system by focusing on edge states and boundary condition. Without the time reversal symmetry breaking, in-gap edge states appear for the fixed boundary

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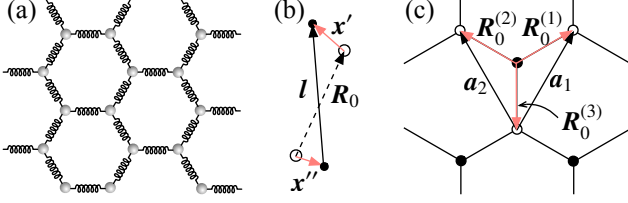


FIG. 1. (a) Schematic picture of the mechanical graphene. (b) Definitions of \mathbf{R}_0 , \mathbf{x}' , and \mathbf{x}'' . (c) The unit vectors \mathbf{a}_1 and \mathbf{a}_2 , and the vectors connecting the nearest neighbor sites, $\mathbf{R}_0^{(i)}$ ($i = 1, 2, 3$).

condition, while they do not appear for the free boundary condition. In the topological view point, symmetry breaking at the edge is behind this contrast.

II. MODEL AND METHODS

The model treated in this paper is composed of mass points, springs, and additional potential energy sources, where the mass points are aligned in the two-dimensional honeycomb lattice. [See Fig. 1(a).] Motion of the mass points is also restricted in the two dimensional space, i.e., the out of plane motion is prohibited. Parameters characterizing this model are mass of the mass points m , a spring constant κ , distance between the nearest neighbor mass points R_0 , and natural length of the springs l_0 . In addition, there are some parameters to characterize additional potential energy other than the one from springs connecting the mass points. The mass m is fixed to 1 for simplicity from now on. $\eta \equiv R_0/l_0$ is not necessarily unity, but when it deviates from unity, each string gives finite force even in equilibrium, which means that we should make the system to avoid shrinkage as a whole. For instance, we need to support the system by appropriate choice of the boundary condition. Or, the potential source other than the springs need to be added. As far as the shrinkage is prevented, the mass points are stable at the equilibrium position since the forces from three springs connected to one mass point cancel out. Dynamical variables for this model is written as $\mathbf{x}_{\mathbf{R}a} = {}^t(x_{\mathbf{R}a}, y_{\mathbf{R}a})$, where \mathbf{R} denotes a lattice point, and a is a sublattice index. As we will see below, the model has commonality with the nearest neighbor tight-binding model of real graphene, but the major difference is that the mass point allows to oscillate in two directions, x and y . There is also the sublattice degrees of freedom, which means that the frequency spectrum for this model has four bands.

For this model, elastic energy of the spring for a given pattern of mass point configuration has a special importance. We first see it for a pair of mass points, introducing parameters \mathbf{R}_0 , \mathbf{x}' , and \mathbf{x}'' as shown in Fig. 1(b). Here, \mathbf{R}_0 is a vector connecting two neighboring mass points in equilibrium. In this paper, we adopt an as-

sumption that all springs have good linearity, i.e., the energy of the spring U_s is always written as

$$U_s = \frac{1}{2}\kappa(l - l_0)^2, \quad (1)$$

where l is the length of the spring at that moment, independent of the value of l . Then, up to the second order in $\delta\mathbf{x} \equiv \mathbf{x}' - \mathbf{x}''$, which is required to investigate harmonic oscillation around the equilibrium alignment, we have

$$U_s = \frac{1}{2}\kappa((R_0 - l_0)^2 + 2(R_0 - l_0)\hat{\mathbf{R}}_0 \cdot \delta\mathbf{x} + \delta x_\mu \gamma_{\hat{\mathbf{R}}_0}^{\mu\nu} \delta x_\nu) \quad (2)$$

with

$$\gamma_{\hat{\mathbf{R}}_0}^{\mu\nu} = (1 - \eta)\delta^{\mu\nu} + \eta\hat{R}_0^\mu \hat{R}_0^\nu, \quad (3)$$

and $\hat{\mathbf{R}}_0 = \mathbf{R}_0/|\mathbf{R}_0|$. Summation over indices appearing as a pair is assumed ($\mu, \nu = x, y$). This formula indicates that when $\eta = 1$, U_s strongly depends on the angle between \mathbf{R}_0 and $\delta\mathbf{x}$, while when $\eta = 0$, U_s is independent of the direction of $\delta\mathbf{x}$. For $\eta = 1$, no force is given by a spring if $\delta\mathbf{x}$ is normal to \mathbf{R}_0 , since such a motion only leads to the rotation of the spring and does not increase the elastic energy in the lowest order approximation. On the other hand, for $\eta = 0$, there is finite force even for $\delta\mathbf{x} = 0$, and then the motion with $\delta\mathbf{x} \perp \mathbf{R}_0$ can increase the energy since there is a finite component of force normal to \mathbf{R}_0 for such motion. In short, the value of η controls a ratio of strength of the restoring force for $\delta\mathbf{x} \parallel \mathbf{R}_0$ and $\delta\mathbf{x} \perp \mathbf{R}_0$. In other words, it controls the relation between the longitudinal and the transverse wave modes.

Then the Lagrangian of this system becomes

$$\mathcal{L} = T - V_1 - V_2 - V_3, \quad (4)$$

with

$$T = \frac{1}{2} \sum_{\mathbf{R}a} \dot{x}_{\mathbf{R}a}^\mu \dot{x}_{\mathbf{R}a}^\mu, \quad (5)$$

$$V_1 = \frac{1}{2}\kappa^0 \sum_{\mathbf{R}a} x_{\mathbf{R}a}^\mu x_{\mathbf{R}a}^\mu, \quad (6)$$

$$V_2 = \frac{1}{2}\kappa \sum_{\langle \mathbf{R}'a\mathbf{R}b \rangle} (x_{\mathbf{R}'a}^\mu - x_{\mathbf{R}b}^\mu) \gamma_{\hat{\mathbf{R}}'_a - \hat{\mathbf{R}}_b}^{\mu\nu} (x_{\mathbf{R}'a}^\nu - x_{\mathbf{R}b}^\nu), \quad (7)$$

$$V_3 = \kappa \sum_{\langle \mathbf{R}'a\mathbf{R}b \rangle} \Delta_{\hat{\mathbf{R}}'_a - \hat{\mathbf{R}}_b}^\mu (x_{\mathbf{R}'a}^\mu - x_{\mathbf{R}b}^\mu), \quad (8)$$

where $\hat{\mathbf{R}}_a$ is a position of the sublattice a associated with the lattice point at \mathbf{R} , and $\Delta_{\hat{\mathbf{R}}}^\mu = (R_0 - l_0)R^\mu/|\mathbf{R}|$. Summation over $\langle \mathbf{R}'a\mathbf{R}b \rangle$ means that we pick up pairs of the nearest neighbor mass points. The term T represents a kinetic energy. The term V_1 is contributed from the potential energy other than the springs connecting the mass points, expanded in a series of $\mathbf{x}_{\mathbf{R}a}$ up to the second order. For simplicity, we assume that this potential is isotropic and sublattice independent. The terms V_2 and

V_3 come from the elastic energy of the springs. As far as the equilibrium is achieved for $\mathbf{x}_{\mathbf{R}a} = 0$, the term V_3 is identically zero, and does not contribute to the equation of motion. Assuming the periodic boundary condition and introducing $u_{\mathbf{k}a}^\mu$ as

$$x_{\mathbf{R}a}^\mu = \frac{1}{N} \sum_{\mathbf{k}} e^{i\mathbf{k} \cdot \mathbf{R}} u_{\mathbf{k}a}^\mu, \quad (9)$$

the Lagrangian is rewritten as

$$\mathcal{L} = \frac{1}{N} \sum_{\mathbf{k}} \mathcal{L}_{\mathbf{k}} \quad (10)$$

$$\mathcal{L}_{\mathbf{k}} = \frac{1}{2} \sum_a \dot{u}_{\mathbf{k}a}^\mu \dot{u}_{-\mathbf{k}a}^\mu - \frac{1}{2} \sum_{ab} \Gamma_{ab}^{\mu\nu} u_{\mathbf{k}a}^\mu u_{-\mathbf{k}b}^\nu, \quad (11)$$

where $\Gamma_{ab}^{\mu\nu} = (\hat{\Gamma}(\mathbf{k}))_{a\mu; b\nu}$. Here,

$$\hat{\Gamma}(\mathbf{k}) = (\kappa^0 + 3\kappa(1 - \frac{\eta}{2}))\hat{1} + \begin{pmatrix} \hat{0} & \hat{\Gamma}_{AB}(\mathbf{k}) \\ \hat{\Gamma}_{AB}(-\mathbf{k}) & \hat{0} \end{pmatrix}, \quad (12)$$

and

$$\hat{\Gamma}_{AB}(\mathbf{k}) = -\kappa(\hat{\gamma}_3 + e^{-i\mathbf{k} \cdot \mathbf{a}_1} \hat{\gamma}_1 + e^{-i\mathbf{k} \cdot \mathbf{a}_2} \hat{\gamma}_2), \quad (13)$$

with

$$\hat{\gamma}_1 \equiv \hat{\gamma}_{\mathbf{R}_0^{(1)}} = (1 - \eta) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \eta \begin{pmatrix} \frac{3}{4} & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{1}{4} \end{pmatrix}, \quad (14)$$

$$\hat{\gamma}_2 \equiv \hat{\gamma}_{\mathbf{R}_0^{(2)}} = (1 - \eta) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \eta \begin{pmatrix} \frac{3}{4} & -\frac{\sqrt{3}}{4} \\ -\frac{\sqrt{3}}{4} & \frac{1}{4} \end{pmatrix}, \quad (15)$$

$$\hat{\gamma}_3 \equiv \hat{\gamma}_{\mathbf{R}_0^{(3)}} = (1 - \eta) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \eta \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \quad (16)$$

[See Fig. 1(c) for the definitions of $\mathbf{R}_0^{(i)}$.]

Now equation to be solved becomes

$$\ddot{u}_{\mathbf{k}a}^\mu + \sum_b \Gamma_{ab}^{\mu\nu}(\mathbf{k}) u_{\mathbf{k}b}^\nu = 0. \quad (17)$$

We seek for the solution periodic in time by introducing a new dynamical variable $\phi_{a\mu}(\mathbf{k})$ as $u_{\mathbf{k}a}^\mu = e^{i\omega t} \phi_{a\mu}(\mathbf{k})$. Then, what we have to solve is

$$-\omega^2 \phi_{a\mu}(\mathbf{k}) + \sum_b \Gamma_{ab}^{\mu\nu}(\mathbf{k}) \phi_{b\nu}(\mathbf{k}) = 0, \quad (18)$$

and to have a nontrivial solution, we must have

$$\text{Det}[-\omega^2 \hat{1} + \hat{\Gamma}(\mathbf{k})] = 0. \quad (19)$$

Practically, this is done by diagonalizing $\hat{\Gamma}(\mathbf{k})$. In that sense, it is possible to map the problem to a quantum mechanical one, by regarding $\hat{\Gamma}(\mathbf{k})$ as a quantum Hamiltonian.

Here, we make a comment on the symmetry of $\hat{\Gamma}(\mathbf{k})$. Except the constant diagonal elements, $\hat{\Gamma}(\mathbf{k})$ has no term

connecting a same kind of sublattice components. Since the constant diagonal elements only add a constant contribution to ω^2 , we say that $\hat{\Gamma}(\mathbf{k})$ has a “chiral” symmetry, since $\hat{\Gamma}(\mathbf{k})$ anticommute with $\hat{\Upsilon} = \text{diag}(1, 1, -1, -1)$ if the constant diagonal part is subtracted. That is, $\hat{\Gamma}'(\mathbf{k}) = \hat{\Gamma}(\mathbf{k}) - (\kappa^0 + 3\kappa(1 - \frac{\eta}{2}))\hat{1}$ satisfies

$$\{\hat{\Gamma}'(\mathbf{k}), \hat{\Upsilon}\} = 0. \quad (20)$$

In fermionic systems, the chiral symmetry plays important roles in various situations. For instance, it stabilizes Dirac cones in 2D cases [32].

Now, the problem is quite similar to the phonon problem in graphene [33–37]. However, what is essential in the following arguments is to control η in a wide range, which is difficult to realize in actual graphene. Furthermore, the effective model for graphene phonon does not respect the “chiral symmetry”, which plays essential role in considering the topological origin of the edge modes. This is because if the vibrational motion in graphene is modeled using a spring-mass model, it requires springs connecting the next nearest neighbor sites and more. Lastly, our model neglects the motion in perpendicular to the plane of the honeycomb lattice.

III. MANIPULATION OF DIRAC CONES

A. Bulk Frequency Spectrum

Now, let us investigate dispersion relation of ω . Before going to general values of η , we consider two limits $\eta = 0$ and $\eta = 1$. For $\eta = 0$, $\hat{\Gamma}_{AB}(\mathbf{k})$ becomes

$$\hat{\Gamma}_{AB}(\mathbf{k}) = -\kappa(1 + e^{-i\mathbf{k} \cdot \mathbf{a}_1} + e^{-i\mathbf{k} \cdot \mathbf{a}_2}) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (21)$$

Then ω^2 looks exactly like a dispersion relation of the nearest neighbor tight-binding model for graphene (NN-graphene) but with extra double degeneracy. In $\eta = 0$ limit, there is no distinction between the longitudinal and transverse wave mode, and in this case, oscillation in x - and y -direction decouple, which leads to doubled NN-graphene dispersion. On the other hand for $\eta = 1$, $\hat{\Gamma}_{AB}(\mathbf{k})$ becomes

$$\hat{\Gamma}_{AB}(\mathbf{k}) = -\kappa \begin{pmatrix} \frac{3}{4}(e^{-ik_1} + e^{-ik_2}) & \frac{\sqrt{3}}{4}(e^{-ik_1} - e^{-ik_2}) \\ \frac{\sqrt{3}}{4}(e^{-ik_1} - e^{-ik_2}) & \frac{1}{4}(e^{-ik_1} + e^{-ik_2}) + 1 \end{pmatrix} \quad (22)$$

with $k_i = \mathbf{k} \cdot \mathbf{a}_i$. Then, as shown in Ref. 21, the second and third bands of ω^2 , dispersion is same as NN-graphene, but the first and the fourth bands are exactly flat and stick to the top and bottom of the other bands. Interestingly, $\hat{\Gamma}(\mathbf{k})$ at $\eta = 1$ is essentially same as the Hamiltonian of p -orbital honeycomb optical lattice model introduced in Ref. 27.

Now, we follow the evolution of the dispersion relation as η changes from 1 to 0. In the following, we always

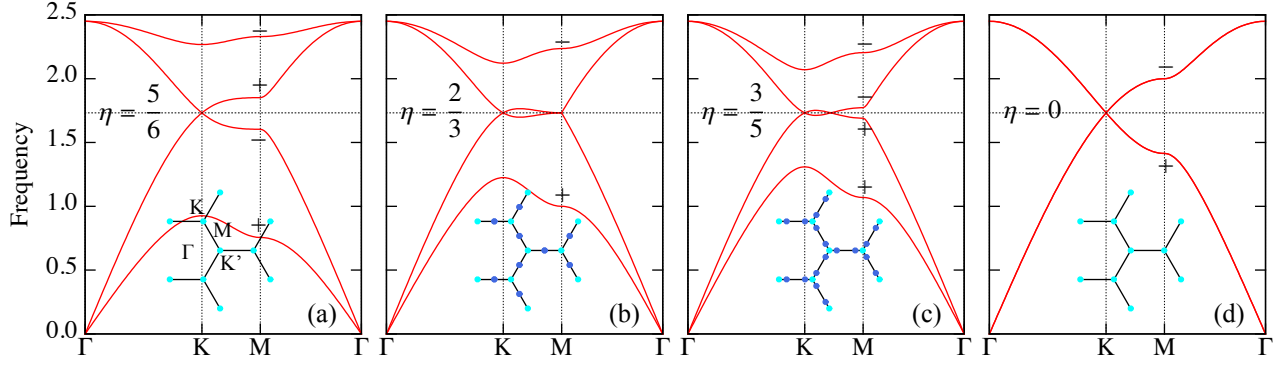


FIG. 2. Dispersion relations for several η . (a) $\eta = 5/6$. (b) $\eta = 2/3$. (c) $\eta = 3/5$. (d) $\eta = 0$. Insets show the positions of the Dirac cones in the Brillouin zone. Cyan and blue dots represent the Dirac cones. + and - signs on the M-point are eigenvalues of the inversion operator Eq. (26). The spring constant κ is scaled as $\kappa = \kappa_0/(1 - \eta/2)$.

scale the spring constant κ as $\kappa = \kappa_0/(1 - \eta/2)$ with $\kappa_0 = 1$, in order to make the total band width of the frequency dispersion constant. This scaling also makes the frequency of the Dirac point constant. The dispersion relation for $\eta = 5/6$ is shown in Fig. 2(a). It is similar to the one for $\eta = 1$ and the Dirac cone is still there at the K-point (and K'-point), but the first and the fourth bands gain finite dispersion and they are no longer flat. At $\eta = 2/3$, as shown in Fig. 2(b), the gap at the M-point in between the second and the third bands closes. In the vicinity of the M-point, the dispersion is linear in Γ -M direction but quadratic in M-K direction. Such a linear vs quadratic dispersion relation is often characteristic for merging of a pair of Dirac cones [25]. In fact, soon after η becomes less than $2/3$, there appear two new Dirac cones, other than those at the K- and K'- points, on M-K lines. [See Fig. 2(c).] Due to the three fold rotational symmetry, there appear six new Dirac cones in the whole Brillouin zone in total. The new Dirac cones move from the M-point to the K-point as η approaches to zero. For $\eta < 2/3$, especially for small η where the positions of the new Dirac cones get close to the K- and K'-points, the dispersion relation looks like the one for bilayer graphene with the trigonal warping [28]. Finally, at $\eta = 0$, the new Dirac cones are absorbed to the originally existing Dirac cones at the K- and K'-points, and the dispersion becomes degenerated NN-graphene dispersion.

The transition at $\eta = 2/3$ that is associated with the Dirac cone merging can be characterized by the ‘‘Herring number’’. For electronic systems, the Herring number is defined as [38]

$$N_H = \frac{1}{2} \sum_{r=1}^{2^d} [N^+(\mathbf{k}_r) - N^-(\mathbf{k}_r)], \quad (23)$$

where d is spatial dimension, \mathbf{k}_r s are the time reversal invariant momenta, and $N^\pm(\mathbf{k}_r)$ is the number of occupied states with parity eigenvalue ± 1 at \mathbf{k}_r . This number is also used by Fu and Kane to obtain Z_2 topological number of the inversion symmetric topological insulator

with the spin-orbit coupling, or to check the existence of Dirac cones without the spin-orbit coupling [39]. For 2D cases, parity of N_H determines the parity of the number of Dirac cones in half of the Brillouin zone as explained below. In Ref. 40, in order to detect Dirac cones in generic 2D systems, we have used the Berry phase defined as

$$i\theta(k_{\parallel}) = \sum_{n \in \text{filled}} \int_{-\pi}^{\pi} dk_{\perp} \langle u_{nk_{\parallel}k_{\perp}} | \nabla_{k_{\perp}} | u_{nk_{\parallel}k_{\perp}} \rangle, \quad (24)$$

where k_{\parallel} and k_{\perp} are two momenta in independent directions, and $|u_{nk_{\parallel}k_{\perp}}\rangle$ is a Bloch wave function [40, 41]. When the system respects both of the time reversal and spatial inversion symmetry, $\theta(k_{\parallel})$ is quantized into 0 or π for the spinless case [40]. As far as the quantization is kept, $\theta(k_{\parallel})$ has to show a jump when it changes as a function of k_{\parallel} , but the jump should be associated with a singularity of the band structure that is nothing more than a Dirac cone. Having this in mind, we can say that if $\theta(0) \neq \theta(\pi) \bmod 2\pi$, odd number of Dirac cones exist in the region $0 < k_{\parallel} < \pi$, i.e., half of the Brillouin zone, while if $\theta(0) = \theta(\pi) \bmod 2\pi$, the number of Dirac cones in the half Brillouin zone is even. On the other hand, the inversion symmetry gives us a relation [39, 42]

$$e^{i(\theta(\pi) - \theta(0))} = \prod_{r=1}^4 (-1)^{N^-(\mathbf{k}_r)} = (-1)^{N_H}. \quad (25)$$

Therefore, the Herring number has an ability to detect the number of Dirac cones. This idea has been applied to the 2D organic Dirac fermion systems [42, 43]. Now, we are handling a classical mechanical system, not an electronic system, but the Herring number is still useful. We have seen that the number of Dirac points in the half Brillouin zone is odd (one) in $\eta > 2/3$ and even (four) in $\eta < 2/3$. This difference should be captured by the Herring number. In Figs. 2(a)-2(d), the eigenvalues of

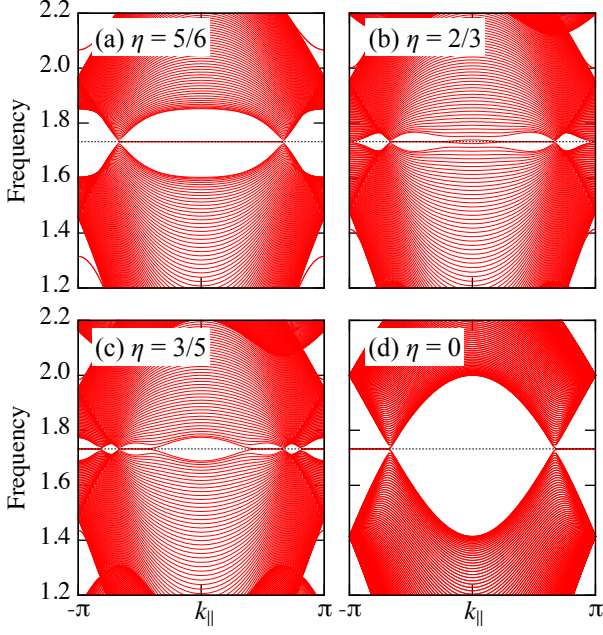


FIG. 3. Edge spectra as a function of the momentum along the edge for several $\eta = 1$ with the fixed boundary condition. (a) $\eta = 5/6$. (b) $\eta = 2/3$. (c) $\eta = 3/5$. (d) $\eta = 0$.

the inversion operator \hat{U}_I , which is defined as

$$\hat{U}_I = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad (26)$$

at the M-point are indicated for each band. We notice that at the transition, $\eta = 2/3$, the positive and negative eigenvalues at the M-point are interchanged. Since there are three distinct M-points in the Brillouin zone, this interchange results in the parity change of the Herring number. Note that Herring originally considers only three-dimensional cases, and in his paper, N_H is derived to detect the number of degeneracy loops in 3D Brillouin zone, instead of Dirac cones.

B. Edge States and Symmetry Protection

Next, we investigate edge states. In this case, we only apply the Fourier transformation in one direction and for the other direction, real space representation is used. Then, we obtain the equation similar to Eq. (19), but with larger number of components in the matrices and vectors. In the following, we concentrate on the zigzag edge. In addition, we use the fixed boundary condition, that is, the mass points at the boundary are connected to the springs that are fixed to the wall, instead of simply removing the springs at the boundary. [See Figs. 4(b) and 4(c).] Importance of the boundary condition is discussed later. The obtained frequency spectra for several

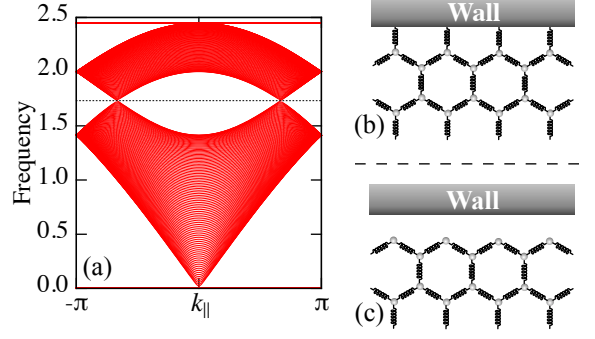


FIG. 4. (a) Edge spectrum for the free boundary condition with $\eta = 1$. (b, c) Schematic pictures of the (b) fixed and (c) free boundary conditions.

values of η as functions of $k_{||}$, momentum parallel to the edge, are shown in Figs. 3(a)-3(d). For $\eta = 5/6$, as in the case of $\eta = 1$ [21], we observe in-gap edge modes near $k_{||} = 0$. The edge modes forms a flat band, since the “chiral symmetry” is preserved in this case. At $\eta = 0$, the system exactly inherits the property of NN-graphene. For instance, the edge modes are found near $k_{||} = \pi$ instead of $k_{||} = 0$ found for $\eta > 2/3$. In short, the position of the edge modes in the edge Brillouin zone is switched from near $k_{||} = 0$ to $k_{||} = \pi$ as η changes from 1 to 0. We observe that the new Dirac cones emerged from the M-points bring new edge modes near $k_{||} = \pi$ and take away the originally existing edge modes near $k_{||} = 0$. Again, note that our model does not necessarily leads to the results exactly same as the case of the actual graphene phonon problems [44, 45].

To see the importance of the boundary condition, ω for $\eta = 1$ obtained with the free boundary condition, where the springs at the boundary are simply removed, is shown in Fig. 4. Note that $\eta \neq 1$ does not go well with the free boundary condition since the finite tension results in the deformation of the equilibrium lattice as the springs at the boundary becomes absent. Then, we have to take the deformation into account, but instead of doing that, we focus on $\eta = 1$. Figure 4 shows that there is no edge mode found as in-gap states. This difference between the two boundary conditions is deeply related to symmetry protection of topological phases, which is one of the most important concept in modern study of topological phenomena. In the real graphene case, it is possible to think a topological origin of the edge modes [4]. For that, the chiral symmetry plays an important role in defining the bulk topological number, the quantized Berry phase, and protecting in-gap edge states. For the fixed boundary condition, the “chiral symmetry”, Eq. (20), is preserved even after the boundary is introduced, since the springs connected to the mass points at the boundary is left. On the other hand, the free boundary condition does not respects the “chiral symmetry” since the absence of the springs at the boundary leads to the *nonuniform* diago-

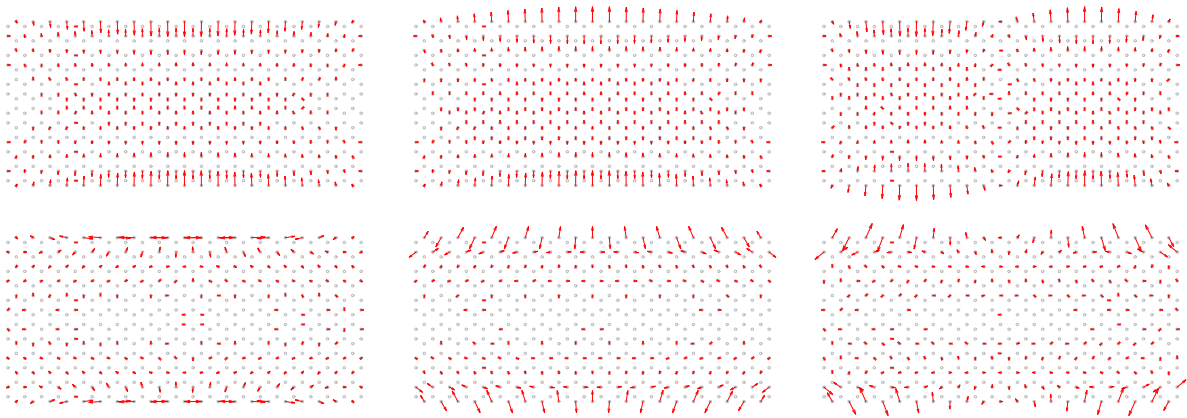


FIG. 5. Real space picture of the typical eigenmodes that are localized to the edge. Upper panels: $\eta = 1$. Lower panels: $\eta = 1/3$. For $\eta = 1$ ($\eta = 1/3$), the neighboring mass points at the edge oscillate in-phase (anti-phase).

nal elements in $\hat{\Gamma}(\mathbf{k})$. This indicates that even the bulk topological property is same, the topological edge states are not necessarily preserved for the free boundary condition, in contrast to the fixed boundary condition, which demonstrates the manifestation of the symmetry protection.

Now, we know that the edge modes for $\eta \sim 1$ and $\eta \sim 0$ has different momenta along the edge. Then, the oscillation pattern in the real space should look different. In order to convince this expectation, we perform a calculation using a rectangular system with long zigzag edges and short armchair edges. This calculation also gives a result to the situation that is easily accessed in experiment. In Fig. 5, the typical eigenmodes whose frequency is close to $\sqrt{3}$, a frequency of the bulk Dirac points, is shown for $\eta = 1/3$ and $\eta = 1$. For $\eta = 1$, the mass points at the boundary basically oscillate in phase with the neighboring mass points on the edge with large wave length background, reflecting the fact that the edge modes are existing near $k_{\parallel} = 0$ in the momentum space. On the other hand, for $\eta = 1/3$, the neighboring mass points on the edge tend to oscillate anti-phase, reflecting the edge mode position in the momentum space. For $\eta = 1$, the mass points at the boundary oscillate approximately normal to the boundary, while for $\eta = 1/3$, oscillation in parallel to the boundary is also allowed. In Fig. 5, only the typical eigenmodes are shown, but we have confirmed that the above statements are basically valid for the mode near $\omega = \sqrt{3}$. Thus, the oscillation pattern of the eigenmode localized at the boundary can be used to detect the phase transition associated with the Dirac cone merging. This is important since the observation of the edge mode with finite size system may be much easier than measuring the dispersion relation itself in detail.

IV. CHERN NUMBER AND CHIRAL EDGE MODES

Lastly, we consider effects of rotation, which induces time reversal symmetry breaking. Uniform rotation as a whole brings two new terms in the Lagrangian, one is the centrifugal force and the other is the Coriolis force. For simplicity, the centrifugal force is neglected. The centrifugal force is second order in the angular frequency of the rotation Ω , while the Coriolis force is first order in Ω , and the second order term can be neglected in the situation that Ω is regarded as a small quantity. Then, taking account of the Coriolis force only, the equation to be solved becomes [21]

$$\text{Det}[-\omega^2 \hat{1} + 2i\omega \hat{\Omega} + \hat{\Gamma}(\mathbf{k})] = 0, \quad (27)$$

where

$$\hat{\Omega} = \begin{pmatrix} \hat{\Omega}_0 & \hat{0} \\ \hat{0} & \hat{\Omega}_0 \end{pmatrix}, \quad \hat{\Omega}_0 = \begin{pmatrix} 0 & \Omega \\ -\Omega & 0 \end{pmatrix}. \quad (28)$$

In this case it is no longer possible to obtain allowed ω by diagonalizing $\hat{\Gamma}(\mathbf{k})$. Instead, it is possible to obtain allowed ω by explicitly evaluating the determinant analytically, and solving a quartic equation of ω^2 . When κ^0 is large, the equation to be solved approximately becomes

$$\text{Det}[-\omega^2 \hat{1} + 2i\sqrt{\kappa^0} \hat{\Omega} + \hat{\Gamma}(\mathbf{k})] = 0, \quad (29)$$

and diagonalization of $2i\sqrt{\kappa^0} \hat{\Omega} + \hat{\Gamma}(\mathbf{k})$ is sufficient to obtain ω . Interestingly, $2i\sqrt{\kappa^0} \hat{\Omega} + \hat{\Gamma}(\mathbf{k})$ is exactly like the Hamiltonian for p -orbital honeycomb optical lattice in Ref. 27.

In Ref. 21, it is shown that the finite Ω induces a gap at the Dirac point for $\eta = 1$. By solving Eq. (27), it is confirmed that such a gap opening remains at work as η is gradually reduced from 1. At some critical point $\eta = \eta_c$, the gap is closed at the M-point. Then, for η less

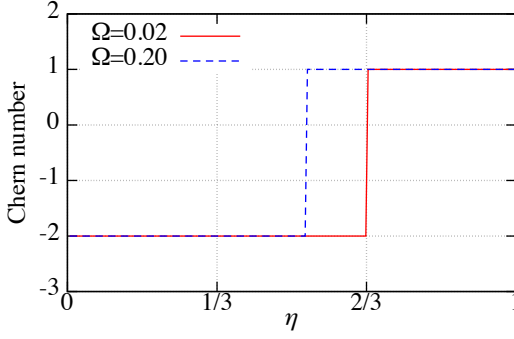


FIG. 6. The Chern number as a function of η for $\Omega = 0.02$ and 0.20 .

than η_c , the system is again fully gapped except $\eta = 0$ where the gap is again absent. η_c depends on Ω , but it is close to $2/3$ in the small Ω limit.

After obtaining ω , eigenmodes are derived as nontrivial solutions for

$$[-\omega^2 \hat{1} + 2i\omega \hat{\Omega} + \hat{\Gamma}(\mathbf{k})]\phi_{n\mathbf{k}} = 0 \quad (30)$$

Then, regarding the solution $\phi_{n\mathbf{k}}$ as a Bloch wave function $|u_{n\mathbf{k}}\rangle$, we can define the Chern number C as

$$C = \sum_{n=1,2} \frac{1}{2\pi} \iint dk_1 dk_2 \left(\frac{\partial \phi_{n\mathbf{k}}^\dagger}{\partial k_1} \cdot \frac{\partial \phi_{n\mathbf{k}}}{\partial k_2} - \frac{\partial \phi_{n\mathbf{k}}^\dagger}{\partial k_2} \cdot \frac{\partial \phi_{n\mathbf{k}}}{\partial k_1} \right) \quad (31)$$

except at $\eta = 0$ and $\eta = \eta_c$ where the gap closing occurs. The summation is taken over $n = 1, 2$, since we are now focusing on the gap between the second and third bands. The Chern number can be used as a topological order parameter for this system. There is a well established method to evaluate this quantity numerically [46]. In Fig. 6, η dependence of the numerically obtained Chern number for $\Omega = 0.02$ and 0.20 is shown. The result at $\eta = 1$ is consistent with the previous work [21]. For small Ω , the transition from Chern number 1 to -2 is observed near $\eta = 2/3$. This is well understood from the fact that each Dirac point carries one-half contribution to the Chern number. For $\Omega = 0.2$, the transition point deviates from $2/3$, but the transition from 1 to -2 still exists. Transition involving change of the Chern number is reported in Ref. 21, but here we propose a transition with completely different mechanism.

In order to observe physical phenomena associated with this transition of the Chern number, we perform a calculation with finite system with a triangular shape having zigzag edges. The Chern number for the mechanical system itself is not a physical observable but the corresponding edge states reflecting nontrivial bulk can be observed experimentally [31]. Here, again, the fixed boundary condition is applied, i.e., the system must be fit in a triangular frame supporting mass points at the boundary by springs. We assume that the system is at rest, or no vibration mode is excited at $t = 0$. Then,

we pick up one of the mass points and apply force that is sinusoidal in time with angular frequency Ω' , i.e., we consider a forced oscillation problem. Now, equation to be solved can be rewritten as

$$\frac{d\mathbf{X}(t)}{dt} = \hat{A}\mathbf{X}(t) + \mathbf{f}(t) \quad (32)$$

where $\mathbf{X}(t) = {}^t(\dot{\mathbf{x}}(t), \mathbf{x}(t))$ and

$$\hat{A} = \begin{pmatrix} 2\hat{\Omega} & -\hat{\Gamma} \\ \hat{1} & 0 \end{pmatrix}. \quad (33)$$

Here, $\hat{\Gamma}$ is a dynamical matrix and $\hat{\Omega}$ is defined as

$$\hat{\Omega} = \begin{pmatrix} \hat{\Omega}_0 & \hat{0} & \cdots \\ \hat{0} & \hat{\Omega}_0 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}. \quad (34)$$

$\mathbf{f}(t)$ denotes an external force, and is explicitly written as $\mathbf{f}(t) = \mathbf{f}_0 \sin \Omega' t$. The elements of \mathbf{f}_0 have a finite value when it corresponds to the picked mass point. A formal solution of this equation is

$$\mathbf{X}(t) = \int_0^t d\tau \exp(\hat{A}(t - \tau)) \mathbf{f}(\tau) \quad (35)$$

when the system is at rest for $t = 0$. Then, $\mathbf{X}(t)$ at any time is evaluated by diagonalizing \hat{A} and perform the integration in Eq. (35).

The selected snapshots of the time evolution for $\eta = 1/3$ (The Chern number is -2) and $\eta = 1$ (The Chern number is 1) calculated with $\Omega = 0.05$ are shown in Fig. 7, where the color map indicates the kinetic energy of each mass point averaged over time period $2\pi/\Omega'_0$ with $\Omega'_0 = \sqrt{3} \sim \Omega'$. In the figure, thick arrows indicates the mass point on which the external force is applied. Here, the direction of force is chosen to be normal to the edge. We use $\Omega' = 1.765$ for $\eta = 1/3$ and $\Omega' = 1.732$ for $\eta = 1$ in order to excite edge modes efficiently. Owing to these choices, for both of $\eta = 1/3$ and $\eta = 1$, the oscillation amplitude is localized near the edge. As in the case of the rectangular system, the neighboring mass points at the edge oscillate anti-phase for $\eta = 1/3$ and in-phase for $\eta = 1$. Following the time evolution of the oscillation, we notice that the oscillation amplitude propagates in the fixed direction, which indicates existence of the chiral edge modes. When the “wave front” of the oscillation reaches to the corner, it goes to the next edge instead of being reflected. For $\eta = 1/3$ with the Chern number -2 , the wave front shows clockwise motion, while for $\eta = 1$ with the Chern number 1, it shows counterclockwise motion. An interesting point of this observation is that the difference between $\eta = 1/3$ and $\eta = 1$ resides in the relation between the transverse and longitudinal modes, and the symmetry of the system is same for $\eta = 1/3$ and $\eta = 1$. Nevertheless, it reverses the direction of the motion of the wave front.

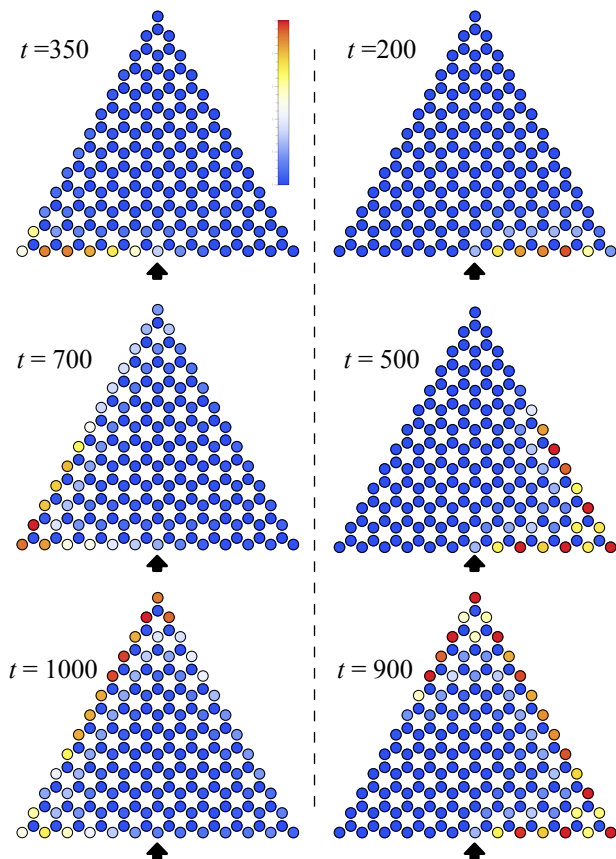


FIG. 7. Snapshots of the time evolution of the system. The color map indicates the kinetic energy of each mass point averaged over the time range $2\pi/\sqrt{3}$. The external force is applied to the mass points indicated by thick arrows. Left panels: $\eta = 1/3$ and $C = -2$. We observe clockwise motion of the edge modes. Right panels: $\eta = 1$ and $C = 1$. We observe counterclockwise motion of the edge modes.

Before closing, we make a comment on the choice of Ω' . In principles, the chiral edge mode is a gapless excitation. However, as we are handling the finite size system, the gap is inevitably open, and we have to be careful on choosing Ω' to excite the edge modes. There are two kinds of finite size effects leading to gap opening. Firstly, the finite size effect simply makes the energy levels discretized, but this does not make much problems in exciting the edge modes. Secondly, in some case, the edge modes localized on different edges are mixed by the finite size effect and lead to a gap that is larger than the one expected from the simple level discretization. In such a case, we have to avoid to place Ω' in such a large gap in order to excite the chiral edge modes effectively. For

$\eta = 1/3$, we encountered with the situation that such a gap is induced, and then, careful choice of Ω' is essential to observe the behavior like Fig. 7.

V. SUMMARY

To summarize, it is shown that the dispersion relation of the mechanical graphene is dramatically changed by controlling only one parameter, the spring tension at equilibrium. When there is no tension at equilibrium, the frequency dispersion is characterized by Dirac cones at the K- and K'-points and flat bands at the bottom and top of the dispersion. As the tension at equilibrium is strengthened, extra Dirac cones are created at the M-points. Then, the newly created Dirac cones migrate from the M-points to the K- or K'-points, and the dispersion becomes similar to the electronic dispersion of the bilayer graphene with the trigonal warping. Eventually, the extra Dirac cones are absorbed to the Dirac cones at the K- and K'-points, and the dispersion is characterized by doubly degenerated Dirac cones in the strong tension limit. The singular bulk dispersion is reflected in the edge spectrum, which is confirmed by a calculation with the ribbon geometry and the fixed boundary condition. Corresponding to the generation and merging of the bulk Dirac cones, the position of the edge states in the momentum space changes, and this change can be detected by observing the real space pattern of the oscillation of the edge modes. On the other hand, the free boundary condition gives no edge modes. In the topological viewpoint, absence of the edge mode for the free boundary condition is explained by breaking of the “chiral symmetry” at the edge. These observation forms a good example of the bulk–edge correspondence. Lastly, Coriolis force is introduced to discuss the time reversal symmetry breaking. Then, generation and merging of the Dirac cones found without Coriolis force appear as a jump in the Chern number, which is defined regarding the vector of normal modes as Bloch wave functions. Furthermore, it is demonstrated that when the sign of the Chern number is flipped, then the propagation direction of the chiral edge modes is really flipped. Importantly, flip in the propagation direction is caused by only controlling the tension at equilibrium, which keeps the symmetry of the system intact. That is, the system looks same, but still the edge modes propagate in the opposite directions.

VI. ACKNOWLEDGMENTS

The work is partly supported by Grants-in-Aid for Scientific Research No. 26247064 from JSPS and No. 25107005 from MEXT.

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